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Nonassociative Geometry of Special Relativity

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The nonassociative axiomatics of the relativistic law of composition of velocities in special relativity is presented. For the first time the canomical unary operations are considered.

1. LOOP

Definition 1.1 (Loop). A set Q together with a binary operation $(\cdot)$ and a two-sided neutral element $\varepsilon \in Q$, $\langle Q, \cdot, \varepsilon \rangle$, is said to be a *loop* if $a \cdot x = b$, $y \cdot a = b$ are uniquely solvable and $\forall q \in Q$, $q \cdot \varepsilon = \varepsilon \cdot q = q$.

Let c be the light velocity in vacuum, and V_c be the velocity space of special relativity,

$$V_c = \{\vec{w} \in \mathbb{R}^3; c > 0, |\vec{w}| < c\}, \qquad \vec{x}, \vec{y} \in V_c, \quad \gamma_{\vec{x}} = \left[1 - \left(\frac{|\vec{x}|}{c}\right)^2\right]^{-1/2}$$

The inner and vector products in $\mathbb{R}^3$ are $(\vec{x} \cdot \vec{y})$ and $\vec{x} \times \vec{y}$. The relativistic law of composition of velocities has the form (Fok, 1955)

$$\vec{x} \boxplus \vec{y} = \frac{\vec{x} + \vec{y}}{1 + (1/c^2)(\vec{x} \cdot \vec{y})} + \frac{1}{c^2} \frac{\gamma_{\vec{x}}}{(1 + \gamma_{\vec{x}})} \frac{\vec{x} \times [\vec{x} \times \vec{y}]}{1 + (1/c^2)(\vec{x} \cdot \vec{y})} \tag{1}$$

$$V_c \ni \vec{x}, \vec{y} \Rightarrow \vec{x} \boxplus \vec{y} \in V_c$$

The law (1) is nonassociative and noncommutative (Nesterov, 1989; Ungar, 1990, 1994, 1997; Sabinin and Miheev, 1993; Sabinin and Nesterov, 1997; Sabinin *et al.*, 1998). We present an axiomatic description of the law (1) for $c = 1$ in the frames of smooth loops and odules.

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Proposition 1.2. For the law of composition of relativistic three-velocities (1) the equations $\vec{a} \boxplus \vec{x} = \vec{b}$, $\vec{y} \boxplus \vec{a} = \vec{b}$ are uniquely solvable. The following identities hold:

$$\vec{0} \boxplus \vec{x} = \vec{x} \boxplus \vec{0} = \vec{x} \qquad \text{existence of two-sided neutral } \vec{0}$$

$$\vec{x} \boxplus [\vec{y} \boxplus (\vec{x} \boxplus \vec{z})] = [\vec{x} \boxplus (\vec{y} \boxplus \vec{x})] \boxplus \vec{z} \qquad \text{left Bol property} \tag{2}$$

$$(\vec{x} \boxplus \vec{y}) \boxplus (\vec{x} \boxplus \vec{y}) = \vec{x} \boxplus (\vec{y} \boxplus (\vec{y} \boxplus \vec{x})) \qquad \text{left Bruck property} \tag{3}$$

Remark 1.3. A loop with the left Bol and left Bruck properties is said to be a *left Bol–Bruck loop*. Therefore $\langle V_1, \boxplus, \vec{0}\rangle$ is a left Bol–Bruck loop. This loop is analytic, since the law (1) is analytic.

Proposition 1.4 (Sabinin, 1981, 1991, 1999). Any C^3-smooth local left Bol–Bruck loop uniquely defines a symmetric space and vice versa.

Proposition 1.5. Let $\sim \vec{a}$ denote the unique solution of $\vec{a} \boxplus \vec{w} = \vec{0}$. For the composition (1), we have

$$\sim(\vec{x} \boxplus \vec{y}) = (\sim \vec{x}) \boxplus (\sim \vec{y}) \qquad \text{automorphic inverse property} \tag{4}$$

$$l(\vec{a}, \vec{b})(\vec{x} \boxplus \vec{y}) = l(\vec{a}, \vec{b})\vec{x} \boxplus l(\vec{a}, \vec{b})\vec{y} \qquad \text{left } A\text{-property}$$

$$L_{\vec{a}}\vec{q} \overset{\text{def}}{=} \vec{a} \boxplus \vec{q}, \qquad l(\vec{a}, \vec{b}) \overset{\text{def}}{=} (L_{(\vec{a} \boxplus \vec{b})})^{-1} \circ L_{\vec{a}} \circ L_{\vec{b}} \tag{5}$$

Remark 1.6 (Sabinin and Sbitneva, 1994; Sabinin, 1999). The left A-property (5) is valid for any left Bol–Bruck loop. For a left Bol loop, the left Bruck property (3) is equivalent to the automorphic inverse property (4).

2. ODULE

Definition 2.1 (Unary operations). Let $\vec{x} \in V_1$, $t \in \mathbb{R}$ and tanh stand for hyperbolic tangent. The unary operations are

$$[t]\vec{x} \overset{\text{def}}{=} (\tanh t \tanh^{-1} |\vec{x}|) \frac{\vec{x}}{|\vec{x}|}, \qquad \vec{x} \neq \vec{0}, \quad [t]\,\vec{0} \overset{\text{def}}{=} \vec{0}, \quad [\mathbb{R}]V_1 = V_1$$

Main Theorem 2.2. Let $t, u \in \mathbb{R}$ and $\vec{x} \in V_1$. Then

$$[t + u]\vec{x} = [t]\vec{x} \boxplus [u]\vec{x} \qquad \text{left monoassociativity} \tag{6}$$

$$[tu]\vec{x} = [t]([u]\vec{x}) \qquad \text{left pseudoassociativity} \tag{7}$$

$$[1_{\mathbb{R}}]\vec{x} = \vec{x} \qquad \text{unitarity} \tag{8}$$

$$[t]\vec{x} \boxplus ([u]\vec{x} \boxplus \vec{y}) = [t + u]\vec{x} \boxplus \vec{y} \qquad \text{left monoalternativity}$$

Definition 2.3 (Odule; Sabinin, 1981, 1999). A loop with unary operations with properties (6)–(8) is called an $\mathbb{R}$-*odule*. An odule with the left Bol and left Bruck properties (2) and (3) is said to be a Bol–Bruck odule.

The loop $\langle V_1, \boxplus, \vec{0}\rangle$ with unary operations, $\forall t \in \mathbb{R}$, $x \to [t]x$, $x \in V_1$, is the Bol–Bruck $\mathbb{R}$-odule $\langle V_1, \boxplus, \vec{0}, ([t])_{t\in\mathbb{R}}\rangle$.

Proposition 2.4 (Sabinin, 1981, 1991, 1999). Any C^3-smooth local left Bol–Bruck loop $\langle Q, \cdot, \varepsilon\rangle$ can be uniquely equipped with smooth unary operations $t \in \mathbb{R}$, $x \mapsto [t]x$, such that $\langle Q, \cdot, \varepsilon, ([t])_{t\in\mathbb{R}}\rangle$ is a left Bol–Bruck odule.

Any left Bol–Bruck odule $\langle Q, \cdot, \varepsilon, ([t])_{t\in\mathbb{R}}\rangle$ is left monoalternative,

$$[t]x \cdot ([u]x \cdot y) = [t + u]x \cdot y$$

When is a C^3-smooth Bol–Bruck odule isomorphic to the left Bol–Bruck odule of relativistic three-velocities?

Proposition 2.5 (Sabinin, 1981, 1991, 1999). Let $\langle Q, \cdot, \varepsilon, ([t])_{t\in\mathbb{R}}$ be a global smooth left Bol–Bruck odule which is not a vector space and there exists, at least locally, near ε, an operation $(x, y) \mapsto x + y$ such that $\langle Q, +, \varepsilon, \mathbb{R}, [,]\rangle$ is a (local) space of dimension 3. If the following pseudolinear identity holds

$$x \cdot y = [\alpha(x, y)]x + [\beta(x, y)]y \quad \alpha(x, y), \beta(x, y) \in \mathbb{R}, \qquad \forall xy \in Q \tag{9}$$

then $\langle Q, \cdot, \varepsilon, ([t])_{t\in\mathbb{R}}\rangle$ up to automorphism is the Bol–Bruck $\mathbb{R}$-odule V_1.

Proof. A slight alternation of the Proof in Theorem 2 in Sabinin and Miheev (1993). ■

Remark 2.6. The law (1) satisfies the pseudolinear identity (9).

Question 2.7. Is it possible to construct a three-dimensional formalism of special relativity on the base of the above nonassociative odule? If so, then one may try to generalize such a construction to general relativity.

3. COMPLEX MODEL

There are other models for the addition of relativistic velocities. The two-dimensional complex model is $D = \{x \in \mathbb{C}; |x| < 1\}$, $\langle D, \boxplus, 0_C\rangle$,

$$x \boxplus y = \frac{x + y}{1 + x^* y} \qquad (x = a + ib;\ x^* = a - ib;\ a, b \in \mathbb{R}) \tag{10}$$

The model $\langle D, \boxplus, 0_C\rangle$ is an analytic Bol–Bruck loop. This model has been used by Ungar (1990, 1994, 1997) to axiomatize the relativistic law of addition of three-velocities. A number of axiomatics have been suggested for (10).

The best known is the gyrogroup (Ungar, 1990, 1994, 1997). Using this concept, Ungar discovered many properties of the relativistic addition of velocities. Sabinin (1995) and Sabinin *et al.* (1998) showed that a gyrogroup is a left Bol–Bruck loop. Also, the close relation of left Bol–Bruck loops with hyperbolic geometry were established.

We have considered for the first time the $\mathbb{R}$-odule, which allows us to give an algebraic presentation of special relativity in the frame of nonassociative algebra.

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